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Asymptotics of the Best Constant in a Certain Markov-Type Inequality

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Let $\|\cdot\|$ be the weighted L^2-norm with Laguerre weight $w(t)=t^\alpha e^{-t}$, $\alpha>-1$. Let P_n be the set of all complex polynomials whose degree does not exceed n, and $\gamma_n(\alpha):=\sup_{p\in P_n}(\|p'\|/\|p\|)$. We show that $\gamma_n(\alpha)/n\to(j_{(\alpha-1)/2,1})^{-1}$ as $n\to\infty$, where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

Key Words: Markov inequality; L^2-norm; Laguerre weight; Pollaczek polynomials; best constant; asymptotics; Bessel functions.

1

Let P_n denote the set of all complex polynomials whose degree does not exceed n. For $p\in P_n$ and arbitrary but fixed $\alpha\in\mathbb{R}$, $\alpha>-1$, let the weighted L^2-norm with (generalized) Laguerre weight be defined by

$$\|p\|:=\left\{\int_0^\infty |p(t)|^2\, t^\alpha e^{-t}\,dt\right\}^{1/2}.$$

In this paper we consider

$$\gamma_n(\alpha):=\sup_{p\in P_n}\frac{\|p'\|}{\|p\|},\qquad n\in\mathbb{N},\tag{1}$$

the best possible constant in the Markov type inequality

$$\|p'\|\leqslant c\,\|p\|,\qquad p\in P_n.$$

If $\alpha=0$, the case of the classical Laguerre weight, $\gamma_n(0)$ is well known.

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In 1944, Schmidt [13] obtained estimates that are asymptotically sharp. Some years later, Turán [14] found the exact value

$$\gamma_n(0) = \left(2 \sin \frac{\pi}{4n+2}\right)^{-1}, \qquad n \in \mathbb{N}.$$

In particular, the above-mentioned results yield

$$\lim_{n \to \infty} \frac{\gamma_n(0)}{n} = \frac{2}{\pi}. \tag{2}$$

For a different proof of (2) see also [5].

If $\alpha > -1$ is arbitrary, little is known. Dörfler [6] presented lower and upper bounds for $\gamma_n(\alpha)$,

$$\frac{n(n+1)}{2(\alpha+1)} - \frac{4(n-1)}{3(\alpha+2)} - \frac{(n-1)(n-2)}{2(\alpha+3)} \leqslant [\gamma_n(\alpha)]^2 \leqslant \frac{n(n+1)}{2(\alpha+1)}, \qquad n \in \mathbb{N}, \tag{3}$$

from which he obtained

$$\frac{1}{\sqrt{(\alpha+1)(\alpha+3)}} \leqslant \liminf_{n \to \infty} \frac{\gamma_n(\alpha)}{n} \leqslant \limsup_{n \to \infty} \frac{\gamma_n(\alpha)}{n} \leqslant \frac{1}{\sqrt{2(\alpha+1)}}. \tag{4}$$

The bounds (3) were derived by a method which had been developed in [4], viewing γ_n as an eigenvalue of a certain matrix. By an analogous method, in [10, pp. 574–576], the following result was achieved: $1/[\gamma_n(\alpha)]^2$ is the smallest zero of the polynomial $T_n(x) = T_n(x, \alpha)$ which is defined recursively by

$$\begin{aligned} &T_{n+1}(x) = (x - d_n)\, T_n(x) - \lambda_n^2 T_{n-1}(x), \qquad n \geqslant 0;\\ &T_{-1}(x) := 0,\ T_0(x) := 1;\\ &d_0 := 1+\alpha,\ d_n := 2 + \frac{\alpha}{n+1},\ n \geqslant 1;\\ &\lambda_0 > 0 \text{ arbitrary},\ \lambda_n^2 := 1 + \frac{\alpha}{n},\ n \geqslant 1. \end{aligned} \tag{5}$$

The $T_n(x, \alpha)$ constitute a system of monic orthogonal polynomials for each $\alpha > -1$ (which is obvious by Favard's theorem [3, p. 21]).

Before presenting our theorem, an interesting related problem should be mentioned. If P_n is restricted to the class W_n of (real) polynomials of exact degree n, all coefficients of which are nonnegative, the analogous problem for $C_n(\alpha) := \sup_{p \in W_n}(\|p'\| / \|p\|)$ was solved by Milovanović [9]

$$C_n(\alpha) = \begin{cases} 1/(2+\alpha)(1+\alpha), & -1 < \alpha \leqslant \alpha_n, \\ n^2/(2n+\alpha)(2n+\alpha-1), & \alpha_n \leqslant \alpha < \infty, \end{cases}$$

where $\alpha_n := \frac{1}{2}(n+1)^{-1}[(17n^2+2n+1)^{1/2} - 3n+1]$. Notice that $C_n(\alpha)$ is bounded for each $\alpha > -1$ as $n \to \infty$ which is rather surprising in view of (4).

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We are ready now to state our theorem.

THEOREM. *Let $\gamma_n(\alpha)$ be defined as in* (1). *Let $j_{\nu,1}$ be the first positive zero of the Bessel function of the first kind of order ν, $J_\nu(z)$. Then*

$$\lim_{n \to \infty} \frac{\gamma_n(\alpha)}{n} = (j_{\frac{\alpha-1}{2},1})^{-1}.$$

Remark 1. Since $j_{\nu,1} = \pi/2$ if $\nu = -1/2$, the theorem yields (2) in the case $\alpha = 0$.

Concerning (4), we would like to mention without proof that $(j_{\nu,1})^{-1}$ is quite close to the arithmetic mean of the bounds given there if $\nu = (\alpha-1)/2$.

Remark 2. It seems to be very difficult to derive the exact value of $\gamma_n(\alpha)$ for arbitrary n and α. The reason is that the knowledge of $\gamma_n(\alpha)$ would imply the knowledge of a zero of certain Pollaczek polynomials—as the proof of our theorem will show. To know such a zero, however, would be a very surprising result.

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The proof of the above theorem requires several preparatory lemmas.

LEMMA 1. *Let $P_n^\lambda(x; a, b)$ be the nth Pollaczek polynomial as defined in* [3, p. 184] *and $T_n(x, \alpha)$ the polynomial as defined in* (5) *and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$T_n(z, \alpha) = (-1)^n \frac{(n+1)}{\alpha} P_{n+1}^{\frac{\alpha}{2}}\left(1 - \frac{z}{2}; -\frac{\alpha}{2}, \frac{\alpha}{2}\right) \tag{6}$$

holds for all $z \in \mathbb{C}$ and $n \geqslant -1$.

Proof. The Pollaczek polynomials $P_n(x) := P_n^\lambda(x; a, b)$ are defined by the recurrence formula [3, p. 184]

$$(n+1)\, P_{n+1}(x) = 2[(a+\lambda+n)\, x+b]\, P_n(x) - (n+2\lambda-1)\, P_{n-1}(x), \qquad n \geqslant 0;$$
$$P_{-1}(x) := 0, \qquad P_0(x) := 1. \tag{7}$$

Inserting for λ, a and b the respective values and substituting $1-x/2$ for x in the above formula, a comparison with (5) leads to (6) for $z = x \in \mathbb{R}$. From this, the assertion follows by analytic continuation. ∎

Remark 3. The choice $\lambda = -a = b = \alpha/2$ of the parameters in $P_n^\lambda(x; a, b)$ causes the corresponding Pollaczek polynomials to be no longer orthogonal. In particular, the nth polynomial $P_n(x)$ has degree $n-1$ for $n \geqslant 1$. This is the reason why in (6) T_n is represented by P_{n+1}.

LEMMA 2. *Let ${}_2F_1(a, b; c; z)$ be the hypergeometric function and $\lambda \in \mathbb{R}$, $\lambda > 0$. Then*

$$P_n^\lambda(\cos w; -\lambda, \lambda) = \frac{(2\lambda)_n}{n!}\, e^{inw}\, {}_2F_1\left(-n, \lambda\left(1+i\tan\frac{w}{2}\right); 2\lambda; 1-e^{-2iw}\right) \tag{8}$$

holds for all $w \in \mathbb{C}$ and $n \geqslant 0$. (If $w = (2m+1)\,\pi$, $m \in \mathbb{Z}$, then the right-hand side of (8) *may be defined by its limit as $w \to (2m+1)\,\pi$.)*

Proof. By [8, p. 462] the polynomials $P_n^\lambda(x; -\lambda+\varepsilon, \lambda)$ are orthogonal if $\varepsilon > 0$ and have the representation

$$P_n^\lambda(\cos\theta; -\lambda+\varepsilon, \lambda) = \frac{(2\lambda)_n}{n!}\, e^{in\theta}\, {}_2F_1(-n, \lambda+it(\theta); 2\lambda; 1-e^{-2i\theta}), \tag{9}$$

where $t(\theta) = [\lambda(1-\cos\theta) + \varepsilon\cos\theta]/\sin\theta$. From (7) we deduce that the left-hand side of (9) depends continuously on ε. Obviously, the same is true for the right-hand side of (9), because ${}_2F_1$ is a finite sum of finite products in ε. So, as $\varepsilon \to 0$, (9) yields (8) for $w = \theta$, $0 \leqslant \theta < \pi$. Finally, analytic continuation gives the desired result for all $w \neq (2m+1)\,\pi$, $m \in \mathbb{Z}$. These points, however, are removable singularities, because the left-hand side of (8) is an entire function. Hence, the limits as $w \to (2m+1)\,\pi$, $m \in \mathbb{Z}$, exist and the right-hand side of (8) must be replaced by the limits in these points. ∎

The following lemma is based on work done by Ismail [8].

Lemma 3. *Let $J_\nu(z)$ be the Bessel function of the first kind of order ν and $\lambda \in \mathbb{R}$, $\lambda > 0$. Then*

$$\lim_{n\to\infty} n^{1-2\lambda} P_n^\lambda\left(\cos\frac{z}{n}; -\lambda, \lambda\right) = \frac{\sqrt{\pi}\, 2^{1-2\lambda}}{\Gamma(\lambda)} \left(\frac{z}{2}\right)^{\frac{1}{2}-\lambda} J_{\lambda-\frac{1}{2}}(z) \tag{10}$$

pointwise for all $z \in \mathbb{C}$.

Proof. If $Re\, a < 1$ and $Re\, c > Re\, b > 0$, the hypergeometric function admits the representation [11, Chap. 5, Sect. 9]

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt \tag{11}$$

for all $z \in \mathbb{C}$. In view of (8) set $a := -n$, $b := \lambda(1 + i\tan(w/2))$, $c := 2\lambda$. Then $Re\, a < 1$ and, obviously, there exists a number $\rho > 0$ such that $Re\, c > Re\, b > 0$ if $|w| < \rho$. Thus, Lemma 2 and (11) yield an integral representation for $P_n^\lambda(\cos w; -\lambda, \lambda)$ if $|w| < \rho$. This representation—with the respective choice of the parameters a and b—is the starting point of the considerations in [8, Chap. 2]. So, we may proceed in a quite similar way as in [8] and omit details here.

First set $t =: (1+v)/2$; this gives

$$\frac{n!\,\Gamma\left(\lambda\left(1+i\tan\frac{w}{2}\right)\right)\Gamma\left(\lambda\left(1-i\tan\frac{w}{2}\right)\right)}{2^{1-2\lambda}\Gamma(2\lambda+n)(\cos w)^n} \times P_n^\lambda(\cos w; -\lambda, \lambda) = A_n(w) + A_n(-w), \tag{12}$$

where

$$A_n(w) := \int_0^1 (1-v^2)^{\lambda-1} \left(\frac{1+v}{1-v}\right)^{i\lambda\tan\frac{w}{2}} (1 - iv\tan w)^n\,dv.$$

Equation (12) holds if $|w| < \rho$.

Let $z \in \mathbb{C}$ be arbitrary but fixed, and replace w by z/n in (12). Since $z/n \to 0$, $\tan(z/2n) \to 0$ and

$$(1 - iv\tan(z/n))^n = \exp[-ivz + \mathcal{O}(n^{-1})]$$

as $n \to \infty$, we conclude that

$$A_n(z/n) \to \int_0^1 (1-v^2)^{\lambda-1}\, e^{-ivz}\,dv$$

and, consequently,

$$A_n(z/n)+A_n(-z/n) \to 2\int_0^1 (1-v^2)^{\lambda-1}\cos(vz)\,dv$$

as $n\to\infty$. This integral can be expressed by a Bessel function [1, 9.1.20]. Thus, we finally arrive at

$$\lim_{n\to\infty}\left[A_n(z/n)+A_n(-z/n)\right]=\sqrt{\pi}\,\Gamma(\lambda)\left(\frac{z}{2}\right)^{\frac{1}{2}-\lambda} J_{\lambda-\frac{1}{2}}(z).$$

As to the left-hand side of (12) (w replaced by z/n), the following asymptotic relations hold as $n\to\infty$: $n!/\Gamma(2\lambda+n)\sim n^{1-2\lambda}$ [1, 6.1.46] and, obviously, $[\cos(z/n)]^n\to 1$.

The combination of these results establishes (10) and completes the proof of the lemma. ∎

The function $(z/2)^{-\nu} J_\nu(z)$ is entire in z for all $\nu\in\mathbb{C}$ [11, p.57]. Thus, in (10) a sequence of entire functions converges pointwise to a limit function which is entire, too. But yet, the convergence of holomorphic functions to a holomorphic limit function is not necessarily uniform on compact subsets [12, Chap. 12, Sect. 3].

In our next lemma we deal with the functions T_n and show the respective uniform convergence on compact subsets of $\mathbb{C}$.

LEMMA 4. *Let $T_n(x,\alpha)$ be the polynomials as defined in* (5) *and* $\alpha\in\mathbb{R}$, $\alpha>0$. *Then*

$$\lim_{n\to\infty}\frac{(-1)^n}{(n+1)^\alpha}T_n\left(4\sin^2\frac{z}{2n+2},\alpha\right)=\frac{\sqrt{\pi}}{2^\alpha\Gamma\left(\frac{\alpha+2}{2}\right)}\left(\frac{z}{2}\right)^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2}}(z) \qquad (13)$$

holds uniformly on compact subsets of the complex z plane.

Proof. Let α be fixed and $T_n(z,\alpha)=: T_n(z)$. From Lemma 1 and the well known identity $2-2\cos w=4\sin^2(w/2)$, $w\in\mathbb{C}$, we deduce that

$$T_n\left(4\sin^2\frac{z}{2n+2}\right)=(-1)^n\frac{(n+1)}{\alpha}P_{n+1}^{\frac{\alpha}{2}}\left(\cos\frac{z}{n+1};-\frac{\alpha}{2},\frac{\alpha}{2}\right) \qquad (14)$$

holds for all $z\in\mathbb{C}$ and $n\geqslant 0$. From this, (13) can be derived easily by the aid of Lemma 3—for the present, the convergence understood as a pointwise one.

Concerning the uniform convergence on compact subsets of $\mathbb{C}$ we show that the respective sequence of functions is uniformly bounded on compact subsets of $\mathbb{C}$. In view of that we first derive a suitable representation of T_n.

Lemma 2 and (14) lead to a representation of T_n by the hypergeometric function ${}_2F_1$. Then, the notation of ${}_2F_1$ as a finite sum [1, 15.4.1] and the two obvious identities

$$\frac{(n+1)(\alpha)_{n+1}}{\alpha(n+1)!}=\frac{(\alpha+1)_n}{n!},\quad \frac{(-n-1)_k}{k!}=\binom{n+1}{k}(-1)^k, \qquad n\geqslant 0,\ 0\leqslant k\leqslant n+1,$$

yield the representation

$$T_n\left(4\sin^2\frac{z}{2n+2}\right)=(-1)^n\frac{(\alpha+1)_n}{n!}e^{iz} \times\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{\left(\frac{\alpha}{2}\left(1+i\tan\frac{z}{2n+2}\right)\right)_k}{(\alpha)_k}(-1)^k\left(1-e^{-\frac{2iz}{n+1}}\right)^k \tag{15}$$

for all $z\in\mathbb{C}$ and $n\geqslant 0$. (As to the points $z=(n+1)(2m+1)\pi$, $m\in\mathbb{Z}$, see Lemma 2.)

Now let K be an arbitrary but fixed compact subset of $\mathbb{C}$ and $z\in K$. Since K is compact, there exists a number $N_1\in\mathbb{N}$ such that $|1+i\tan(z/(2n+2))|\leqslant 2\ \forall z\in K$ and, consequently,

$$\frac{\left|\left(\frac{\alpha}{2}\left(1+i\tan\frac{z}{2n+2}\right)\right)_k\right|}{|(\alpha)_k|}\leqslant 1, \qquad 0\leqslant k\leqslant n+1,$$

if $n>N_1$. (Observe that $\alpha\in\mathbb{R}$, $\alpha>0$ by assumption!) Hence, passing over to the absolute values on either side of (15), the Binomial Theorem can be applied to the sum. Thus, (15) together with the above estimates imply that

$$\left|T_n\left(4\sin^2\frac{z}{2n+2}\right)\right|\leqslant\frac{(\alpha+1)_n}{n!}|e^{iz}|\left(1+\left|1-e^{-\frac{2iz}{n+1}}\right|\right)^{n+1}, \qquad n>N_1. \tag{16}$$

Obviously $|1-e^{-2iz/(n+1)}|\leqslant 2|z|/(n+1)+\mathcal{O}(n^{-2})$ and, since K is compact, there exists a natural number $N\geqslant N_1$ such that $|1-e^{-2iz/(n+1)}|<1\ \forall z\in K$ if $n>N$. Then

$$\begin{aligned}\left(1+\left|1-e^{-\frac{2iz}{n+1}}\right|\right)^{n+1}&=\exp\left[(n+1)\log\left(1+\left|1-e^{-\frac{2iz}{n+1}}\right|\right)\right]\\&\leqslant\exp[2|z|+\mathcal{O}(n^{-1})]\end{aligned}$$

which implies, if $n > N$,

$$|e^{iz}| \, (1+|1-e^{-\frac{2iz}{n+1}}|)^{n+1} \leqslant M_1 \qquad \forall z \in K, \tag{17}$$

M_1 being a constant depending on N and K only.

From the obvious identity

$$\frac{(\alpha+1)_n}{(n+1)^\alpha \, n!} = \frac{1}{\Gamma(\alpha+1)} (n+1)^{-\alpha} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}$$

we deduce by [1, 6.1.46] that

$$\lim_{n \to \infty} \frac{(\alpha+1)_n}{(n+1)^\alpha \, n!} = \frac{1}{\Gamma(\alpha+1)}$$

and, consequently, if $n > N$,

$$\frac{(\alpha+1)_n}{(n+1)^\alpha \, n!} \leqslant M_2, \tag{18}$$

M_2 being a constant depending on N only.

Finally, let M be defined as

$$M := \max_{0 \leqslant n \leqslant N} \left\{ M_1 M_2, \sup_{z \in K} \left| \frac{(-1)^n}{(n+1)^\alpha} T_n \left(4 \sin^2 \frac{z}{2n+2} \right) \right| \right\}.$$

Then, by (16), (17), (18),

$$\left| \frac{(-1)^n}{(n+1)^\alpha} T_n \left(4 \sin^2 \frac{z}{2n+2} \right) \right| \leqslant M \qquad \forall n \geqslant 0, \qquad \forall z \in K,$$

which means that the respective sequence of functions is uniformly bounded on K. Hence, by well known theorems (e.g., [2, p. 166)]), the convergence in (13) is uniform on compact subsets of $\mathbb{C}$ and we are done. ▮

It seems to be difficult to prove Lemma 4 if $\alpha \in (-1, 0]$ by the techniques used so far. Thus we must apply new methods! In particular, the functions T_n and the related expressions have to be viewed as functions in the additional complex variable α.

LEMMA 5. *Let T_n and J_ν be defined as above and $z \in \mathbb{C}$, $\alpha \in \mathbb{C}$, $n \geqslant 0$. Furthermore let*

$$f_n(z, \alpha) := \frac{(n+1)^{-\alpha}(\alpha+1)_n}{n!} e^{iz} \sum_{k=0}^{n+1} \binom{n+1}{k} \times \frac{\left(\frac{\alpha}{2}\left(1+i \tan \frac{z}{2n+2}\right)\right)_k}{(\alpha)_k} (-1)^k \left(1-e^{-\frac{2iz}{n+1}}\right)^k,$$

$$F(z, \alpha) := \frac{\sqrt{\pi}}{2^\alpha \Gamma\left(\frac{\alpha+2}{2}\right)} \left(\frac{z}{2}\right)^{\frac{1-\alpha}{2}} J_{\frac{\alpha-1}{2}}(z).$$

Then, for arbitrary but fixed $n \geqslant 0$,

$$(-1)^n (n+1)^{-\alpha} T_n\left(4 \sin^2 \frac{z}{2n+2}, \alpha\right) = f_n(z, \alpha) \tag{19}$$

for all $z \in \mathbb{C}$ and all $\alpha \in \mathbb{C}$. Moreover, both $f_n(z, \alpha)$ and $F(z, \alpha)$ are entire functions of z for every $\alpha \in \mathbb{C}$ and entire functions of α for every $z \in \mathbb{C}$. ($f_n(z, \alpha)$ has a removable singularity if $z = (n+1)(2m+1)\pi$, $m \in \mathbb{Z}$, or $\alpha = j$, $j \leqslant 0$, $j \in \mathbb{Z}$. In these points $f_n(z, \alpha)$ may be defined by its limit.)

Proof. Equation (15) establishes (19) for all $z \in \mathbb{C}$ and every $\alpha \in \mathbb{R}$, $\alpha > 0$. Since the left-hand side of (19) is obviously entire in z and entire in α (see (5)), (19) follows by analytic continuation; as to the singularities of $f_n(z, \alpha)$, we use the same argument as in the proof of Lemma 2. In particular, $f_n(z, \alpha)$ clearly is entire in z and entire in α.

The assertion concerning $F(z, \alpha)$ follows from [1, 6.1.3; 11, p. 57]. ∎

LEMMA 6. *Let $f_n(z, \alpha)$ be defined as in Lemma 5 and $E := \{\alpha \in \mathbb{C} \mid |\alpha| < 1\}$. Let $K \subseteq \mathbb{C}$ be an arbitrary but fixed compact subset of $\mathbb{C}$. Then there exists a constant $M > 0$ depending on K only such that*

$$|f_n(z, \alpha)| \leqslant M \qquad \forall z \in K, \qquad \forall \alpha \in E, \qquad \forall n \geqslant 0.$$

Proof. Let $z \in K$ and $\alpha \in E$. Since K is compact, there exists a number $N_1 \in \mathbb{N}$ such that $|1 + i \tan(z/(2n+2))| \leqslant 2 \; \forall z \in K$ and, consequently,

$$\frac{\left|\left(\frac{\alpha}{2}\left(1+i \tan \frac{z}{2n+2}\right)\right)_k\right|}{|(\alpha)_k|} \leqslant \begin{cases} 1, & 0 \leqslant k \leqslant 1 \\ k!/(|\alpha+1| \cdots |\alpha+k-1|), & 2 \leqslant k \leqslant n+1, \end{cases} \tag{20}$$

if $n > N_1$.

The obvious inequality $(j+Re\,\alpha)^2 > (j-1)^2$, $j \geqslant 1$, implies $|\alpha+j| > j-1$, $j \geqslant 1$, and hence

$$\frac{k!}{|\alpha+1| \cdots |\alpha+k-1|} \leqslant \frac{k(k-1)}{|\alpha+1|}, \qquad 2 \leqslant k \leqslant n+1,$$

which may be applied to (20).

Using the identity

$$\sum_{k=2}^{n+1} \binom{n+1}{k} k(k-1)\, w^k = (n+1)\, nw^2(1+w)^{n-1}, \qquad w \in \mathbb{C},$$

which is a consequence of the Binomial Theorem, we finally obtain for the sum in $f_n(z, \alpha)$, if $n > N_1$,

$$\left| \sum_{k=0}^{n+1} \right| \leqslant \frac{1}{|\alpha+1|} \left[|\alpha+1| + |\alpha+1|\, (n+1)\, |1-e^{-\frac{2iz}{n+1}}| \right.$$
$$\left. +(n+1)\, n\, |1-e^{-\frac{2iz}{n+1}}|^2\, (1+|1-e^{-\frac{2iz}{n+1}}|)^{n-1} \right]. \tag{21}$$

In the same way as in the proof of Lemma 4 we conclude that $|1-e^{-2iz/(n+1)}| \leqslant 2\,|z|/(n+1) + \mathcal{O}(n^{-2})$ and that $(1+|1-e^{-2iz/(n+1)}|)^{n-1}$ is bounded from above for all $z \in K$ if $n > N_2 \geqslant N_1$. Moreover, $|\alpha+1| < 2$ holds for all $\alpha \in E$. These facts, applied to (21), yield the following estimate if $n > N_2$,

$$|f_n(z, \alpha)| \leqslant M_1 \left| \frac{(n+1)^{-\alpha}\, (\alpha+1)_n}{n!\,(\alpha+1)} \right| \qquad \forall z \in K, \qquad \forall \alpha \in E, \tag{22}$$

M_1 being a constant depending on N_2 and K only.

Next we consider the sequence of functions [11, Chap. 2, 1.3]

$$\Gamma_n(w) := \int_0^n \left(1-\frac{t}{n}\right)^n t^{w-1}\, dt = \frac{n!\, n^w}{w(w+1) \cdots (w+n)}, \qquad n \in \mathbb{N},$$

defined in $H := \{w \in \mathbb{C} \mid Re\, w > 0\}$ and converging to $\Gamma(w)$. Since

$$|\Gamma_n(w)| \leqslant \int_0^n e^{-t} t^{Re\, w-1}\, dt \leqslant \Gamma(Re\, w),$$

the $\Gamma_n(w)$ are uniformly bounded on every compact subset of H and thus $\lim_{n \to \infty} \Gamma_n(w) = \Gamma(w)$ holds uniformly on compact subsets of H.

Let $S := \{w \in \mathbb{C} \mid |w-2| \leqslant 1\} \subseteq H$. Since S is compact and $\Gamma(w) \neq 0\ \forall w \in H$, there exists a real constant $d > 0$ such that $|\Gamma(w)| \geqslant d\ \forall w \in S$. On the other hand, the uniform convergence of the Γ_n on S implies that there exists a number $N \in \mathbb{N}$ (Let $N \geqslant N_2$ in addition!) such that $|\Gamma(w) - \Gamma_n(w)| < d/2$ for all $w \in S$ if $n > N$. Hence it follows that

$$|\Gamma_n(w)| > d/2 \qquad \forall w \in S, \qquad \forall n > N. \tag{23}$$

Let us apply now (23) to our problem!

Since $|\alpha + j| > j - 1$, $j \geqslant 1$ (see above), we obtain by (23) and a short computation

$$\left| \frac{n!\,(\alpha+1)}{(n+1)^{-\alpha}\,(\alpha+1)_n} \right| = |\Gamma_{n+1}(\alpha+2)| \, \frac{|\alpha+n+1|\;|\alpha+n+2|\;|\alpha+n+3|}{(n+1)^3}$$
$$> \frac{d}{2}\, \frac{n(n+1)(n+2)}{(n+1)^3} \geqslant \frac{3d}{8}$$

for all $\alpha \in E$ and $n > N$. Hence it follows that there exists a constant $M_2 > 0$ such that

$$\left| \frac{(n+1)^{-\alpha}\,(\alpha+1)_n}{n!\,(\alpha+1)} \right| < M_2 \qquad \forall \alpha \in E \tag{24}$$

if $n > N$. Let M be defined as

$$M := \max_{0 \leqslant n \leqslant N} \{M_1 M_2, \sup_{\substack{z \in K \\ \alpha \in E}} |f_n(z, \alpha)|\}.$$

Then, by (22) and (24), the assertion of the lemma follows and we are done. ▮

Our last lemma extends Lemma 4 to values of α from the interval $(-1, 0]$.

LEMMA 7. *Let f_n and F be defined as in Lemma 5 (observe also* (19)*) and let $\alpha \in \mathbb{R}$, $\alpha > -1$. Then*

$$\lim_{n \to \infty} f_n(z, \alpha) = F(z, \alpha)$$

holds uniformly on compact subsets of the complex z plane.

Proof. First let $z \in \mathbb{C}$ be arbitrary but fixed. By Lemma 5, the functions $f_n(z, \alpha)$ are holomorphic functions of α in the unit disc E and, by Lemma 6, they are uniformly bounded on compact subsets of E. Moreover, by Lemma 4, the sequence $f_n(z, \alpha)$ is converging on the interval $(0, 1)$. Then, a combination of Montel's theorem and Vitali's theorem [2, p. 166] implies that the functions (of α) $f_n(z, \alpha)$ converge uniformly on compact subsets of E and, consequently, the limit function $G(z, \alpha) := \lim_{n \to \infty} f_n(z, \alpha)$ is a holomorphic function of α in E. On the other hand, by Lemma 5, $F(z, \alpha)$ is a holomorphic function (of α) in E, too. Since $G(z, \alpha) = F(z, \alpha)$ for $\alpha \in (0, 1)$, it thus follows by analytic continuation that $G(z, \alpha) = F(z, \alpha)$ for all $\alpha \in E$. Hence the limit relation $\lim_{n \to \infty} f_n(z, \alpha) = F(z, \alpha)$ holds for all $\alpha \in E$ and therefore, in particular, for $\alpha \in (-1, 0]$. Since $z \in \mathbb{C}$ was arbitrary, we obtain

$$\lim_{n \to \infty} f_n(z, \alpha) = F(z, \alpha) \qquad \forall z \in \mathbb{C}, \qquad \forall \alpha \in (-1, 0]. \tag{25}$$

Now let $\alpha \in (-1, 0]$ be arbitrary but fixed. By Lemma 5, the functions $f_n(z, \alpha)$ are entire functions of z and, by Lemma 6, they are uniformly bounded on compact subsets of the z plane. As in the proof of Lemma 4 we conclude that the convergence in (25) is uniform on compact subsets of the z plane. Together with Lemma 4 this proves the assertion of the lemma. ▮

We are ready now to prove the theorem.

Proof of the Theorem. Let $\alpha \in \mathbb{R}$, $\alpha > -1$ be arbitrary but fixed. First we need some information about the zeros of the functions (of z) $f_n(z, \alpha) =: f_n(z)$ and $F(z, \alpha) =: F(z)$.

Obviously, the zeros of $f_n(z)$ are the zeros of $T_n(z) := T_n(z, \alpha)$ (cf. (19)). Since the $T_n(x)$, $n \geqslant 0$, constitute a system of orthogonal polynomials (cf. (5)), all these zeros are real and simple. Let x_n be the smallest zero of $T_n(z)$. Then $x_n = 1/[\gamma_n(\alpha)]^2 > 0$ (cf. Chapter 1).

The representation of $J_{(\alpha-1)/2}(z)$ as a series [11, p. 57] shows that $F(0) \neq 0$. Consequently, $F(z) \neq 0$ on the interval $[0, j_{(\alpha-1)/2, 1})$; the zero $j_{(\alpha-1)/2, 1}$ is a simple zero of $F(z)$ [1, 9.5].

The following considerations rely on the Theorem of Hurwitz and some of its corollaries.

Lemma 7 and [7, Corollary 4.10e] imply that there exists a sequence $\langle z_n \rangle$ such that $\lim_{n \to \infty} z_n = j_{(\alpha-1)/2, 1}$ and $T_n(4 \sin^2(z_n/(2n+2))) = 0$. Let $y_n := 4 \sin^2(z_n/(2n+2))$. Since all the zeros of $T_n(z)$ are real and x_n is the smallest one, we conclude that $0 < x_n \leqslant y_n$.

Clearly, the convergent sequence $\langle z_n \rangle$ is bounded. Hence there exists a number $N_1 \in \mathbb{N}$ such that $|z_n/(2n+2)| < \pi/2$ if $n > N_1$. This implies that $z_n \in \mathbb{R}$ if $n > N_1$, since $\sin(z_n/(2n+2)) \in \mathbb{R}$. Moreover, let N_1 be sufficiently large such that $0 < z_n/(2n+2) < \pi/2$ if $n > N_1$. (Observe that $z_n \to j_{(\alpha-1)/2, 1} > 0$ as $n \to \infty$!)

Assume that $x_n < y_n$ for infinitely many $n > N_1$, say $n = n_j$, $j \in \mathbb{N}$. Then there exists a sequence $\langle z'_{n_j} \rangle$ such that $x_{n_j} = 4 \sin^2(z'_{n_j}/(2n_j + 2))$ and

$0 < z'_{n_j} < z_{n_j}\ \forall j \in \mathbb{N}$. Since $\langle z_{n_j} \rangle$ is bounded, there exists a point of accumulation for the sequence $\langle z'_{n_j} \rangle$, say z_0. This point obviously has the properties $0 \leqslant z_0 \leqslant j_{(\alpha-1)/2,1}$ and, by [7, Theorem 4.10d], $F(z_0) = 0$. Since $F(z)$ does not vanish on the interval $[0, j_{(\alpha-1)/2,1})$, z_0 cannot be smaller than $j_{(\alpha-1)/2,1}$. But if $z_0 = j_{(\alpha-1)/2,1}$, then every neighbourhood of $j_{(\alpha-1)/2,1}$ contains at least two different zeros of $T_{n_j}(4\sin^2(z/(2n_j+2)))$, namely z_{n_j} and z'_{n_j}. This implies, by [7, Theorem 4.10d], that the zero $j_{(\alpha-1)/2,1}$ of $F(z)$ cannot be simple. Contradiction! Thus we have to drop the above assumption and we conclude that there exists a natural number $N > N_1$ such that $x_n = y_n$ for all $n > N$. Then, if $n > N$,

$$x_n = 4\sin^2\frac{z_n}{2n+2} = \frac{z_n^2}{(n+1)^2} + \mathcal{O}(n^{-4})$$

which implies

$$\lim_{n\to\infty} n^2 x_n = \lim_{n\to\infty} z_n^2 = (j_{\frac{\alpha-1}{2},1})^2.$$

Since $x_n = 1/[\gamma_n(\alpha)]^2$, we are done. ∎

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